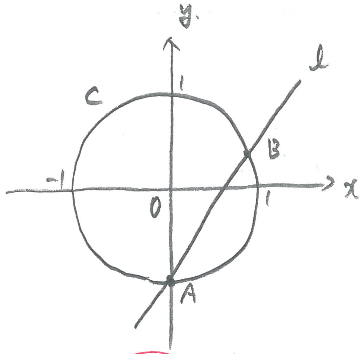
第7回

[I].

[1]

$$\begin{cases} \ell:\ y=(\tan\theta)x-1 & \text{—①} \\ C:\ x^2+y^2=1 & \text{—②} \\ (0<\theta<\frac{\pi}{2}) \end{cases}$$



$A(0,\ \underline{-1})$ アイ, $B(p,\ q)$

①より $q=(\tan\theta)p-1.$

②より $p^2+q^2=1.$

代入.

$$p^2+\{(\tan\theta)p-1\}^2=1$$

$$p^2+(\tan^2\theta)\cdot p^2-2(\tan\theta)p+1=1$$

$$(1+\tan^2\theta)p^2-2(\tan\theta)p=0.$$

$$p\{(1+\tan^2\theta)p-2\tan\theta\}=0$$

$$p=0,\ \frac{2\tan\theta}{1+\tan^2\theta}$$

$p\neq 0$ より. $p=\dfrac{2\tan\theta}{1+\tan^2\theta}$ ウエ.

$1+\tan^2\theta=\dfrac{1}{\cos^2\theta}$ オ より

$$p=\frac{2\cdot\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}=2\cdot\frac{\sin\theta}{\cancel{\cos\theta}}\cdot\cos^2\theta$$

$$=2\sin\theta\cos\theta$$ カ.

$$q=\tan\theta\cdot(2\sin\theta\cos\theta)-1$$

$$=\frac{\sin\theta}{\cancel{\cos\theta}}\cdot 2\sin\theta\cancel{\cos\theta}-1=2\sin^2\theta-1$$ キク.

よって.

$$p+q=2\sin\theta\cos\theta+2\sin^2\theta-1$$

$$=2\sin\theta\cos\theta-(1-2\sin^2\theta)$$

$$=\sin 2\theta-\cos 2\theta$$

$$=\sqrt{2}\sin\left(2\theta-\frac{\pi}{4}\right)$$ ケ コ.

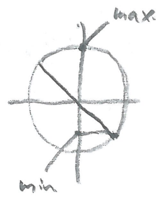
$0<\theta<\frac{\pi}{2}$ より. $0<2\theta<\pi$

$$-\frac{\pi}{4}<2\theta-\frac{\pi}{4}<\frac{3}{4}\pi.$$

$$-\frac{1}{\sqrt{2}}<\sin\left(2\theta-\frac{\pi}{4}\right)\leqq 1.$$

$$-1<\sqrt{2}\sin\left(2\theta-\frac{\pi}{4}\right)\leqq\sqrt{2}.$$

$$\underline{-1}<p+q\leqq\underline{\sqrt{2}}$$ サシ ス.



$$pq=2\sin\theta\cos\theta\cdot(2\sin^2\theta-1)$$

$$=2\sin\theta\cos\theta\cdot\{-(1-2\sin^2\theta)\}$$

$$=\sin 2\theta\cdot(-\cos 2\theta)=-\sin 2\theta\cos 2\theta$$

$$=-\frac{1}{2}\cdot 2\sin 2\theta\cdot\cos 2\theta=-\frac{1}{2}\sin 4\theta.$$

$0<\theta<\frac{\pi}{2}$ より. $0<4\theta<2\pi.$

$$-1\leqq\sin 4\theta\leqq 1.\qquad -\frac{1}{2}\leqq-\frac{1}{2}\sin 4\theta\leqq\frac{1}{2}$$

$$-\frac{1}{2}\leqq pq\leqq\frac{1}{2}$$

pq が最大値をとるのは.

$-\frac{1}{2}\sin 4\theta=\frac{1}{2}\quad \sin 4\theta=-1.\quad 4\theta=\frac{3}{2}\pi.$

すなわち. $\theta=\dfrac{3}{8}\pi$ セソ のときで

最大値は $\dfrac{1}{2}$ タチ

[2].

$1 - \log_{10} 2 \leqq \log_{10}(3^m - 2) + \log_{10}(2^n - 3) < 3\log_{10} 2$ —①

(m, n：整数)

真数条件より

$3^m - 2 > 0$, $3^m > 2$. $m \geqq 1$

$2^n - 3 > 0$　$2^n > 3$　$n \geqq 2$

$1 - \log_{10} 2 = \log_{10} 10 - \log_{10} 2 = \log_{10}\frac{10}{2} = \log_{10} 5.$

$\log_{10}(3^m - 2) + \log_{10}(2^n - 3) = \log_{10}(3^m - 2)(2^n - 3)$

$3\log_{10} 2 = \log_{10} 2^3 = \log_{10} 8$　より

$\log_{10} 5 \leqq \log_{10}(3^m - 2)(2^n - 3) < \log_{10} 8$

$5 \leqq (3^m - 2)(2^n - 3) < 8.$

$m = 1$ のとき

$5 \leqq (3-2)(2^n - 3) < 8$

$5 \leqq 2^n - 3 < 8$

$8 \leqq 2^n < 11$　$n = 3$

$m = 2$ のとき

$5 \leqq (9-2)(2^n - 3) < 8$

$5 \leqq 7(2^n - 3) < 8$

$\frac{5}{7} \leqq 2^n - 3 < \frac{8}{7} = 1\frac{1}{7}$

$3\frac{5}{7} \leqq 2^n < 4\frac{1}{7}$　$n = 2$

よって

$(m, n) = (1, 3), (2, 2)$

$(m, n) = (1, 3)$ のとき

$\{(3^m - 2)(2^n - 3)\}^k$

$= \{(3-2)(2^3 - 3)\}^k$

$= (8-3)^k = 5^k$　が　70桁となる k.

$10^{69} \leqq 5^k < 10^{70}$

$\log_{10} 10^{69} \leqq \log_{10} 5^k < \log_{10} 10^{70}$

$69 \leqq k\log_{10}\frac{10}{2} < 70$

$69 \leqq k(\log_{10} 10 - \log_{10} 2) < 70$

$69 \leqq k(1 - 0.3010) < 70$

$69 \leqq k \times 0.6990 < 70.$

$69 \leqq \frac{699}{1000}k < 70.$

$\frac{\cancel{69}^{23}}{\cancel{699}_{233}} \times 1000 \leqq k < \frac{70}{699} \times 1000$

$23.00 \div 233 = 0.098$…　$70.000 \div 699 = 0.100$

$98.\cdots \leqq k < 100.\cdots$

$99 \leqq k \leqq 100.$

k の最小は 99

[2].

$a > -1$.

$f(x) = 2x^3 + 3(1-a)x^2 - 6ax + 1$

$f'(x) = 6x^2 + 6(1-a)x - 6a$

$= 6\{x^2 + (1-a)x - a\}$

$= 6(x+1)(x-a)$ ア〜ウ.

$a > -1$ より 増減表は以下.

x	$\cdots$	-1	$\cdots$	a	$\cdots$
f'	$+$	0	$-$	0	$+$
f	↗	$f(-1)$	↘	$f(a)$	↗

$f(a) = g(a)$

$x = a$ で 極小となり エ

$g(a) = f(a) = 2a^3 + 3(1-a)\cdot a^2 - 6a^2 + 1$

$= 2a^3 + 3a^2 - 3a^3 - 6a^2 + 1$

$= -a^3 - 3a^2 + 1$. オ〜キ.

$g'(a) = -3a^2 - 6a$

$= -3a(a+2)$

a	$\cdots$	-2	$\cdots$	-1	$\cdots$	0	$\cdots$
g'	$-$	0	$+$		$+$	0	$-$
g	↘		↗		↗	$g(0)$	↘

$a = 0$ で $g(a)$ は最大; ク

最大値は, $g(0) = 1$ ケ.

(※ 以下 $a = 0$)

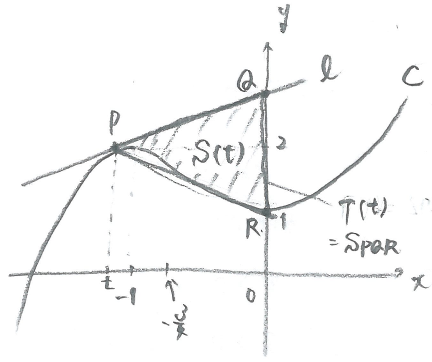
C: $f(x) = 2x^3 + 3x^2 + 1$.

$f'(x) = 6x^2 + 6x$.

x	$\cdots$	-1	$\cdots$	0	$\cdots$
f'	$+$	0	$-$	0	$+$
f	↗	2	↘	1	↗

$f(-1) = -2 + 3 + 1 = 2$.

C上の点Pを通る接線lの式は



$y = f'(t)(x-t) + f(t)$

$y = (6t^2+6t)(x-t) + 2t^3 + 3t^2 + 1$

l: $y = (6t^2+6t)x - 4t^3 - 3t^2 + 1$ コ〜セ

lとy軸との交点Qのy座標qは. $q > 1$より.

$x = 0$ 代入すると.

$-4t^3 - 3t^2 + 1 > 1$.

$4t^3 + 3t^2 < 0$

$t^2(4t+3) < 0$

$t^2 > 0$ より

$4t + 3 < 0$

$4t < -3$

$t < -\frac{3}{4}$ ソ〜チ

(※ 以下 $t < -\frac{3}{4}$)

$S(t) = \int_t^0 \{(6t^2+6t)x - 4t^3 - 3t^2 + 1 - (2x^3 + 3x^2 + 1)\}dx$

$= \int_t^0 \{-2x^3 - 3x^2 + (6t^2+6t)x - 4t^3 - 3t^2\}dx$

$= \left[-\frac{1}{2}x^4 - x^3 + (3t^2+3t)x^2 - (4t^3+3t^2)x\right]_t^0$

$= -\left\{-\frac{1}{2}t^4 - t^3 + (3t^2+3t)t^2 - (4t^3+3t^2)t\right\}$

$= \frac{1}{2}t^4 + t^3 - 3t^4 - 3t^3 + 4t^4 + 3t^3$

$= \frac{3}{2}t^4 + t^3 = \frac{3t^4 + 2t^3}{2} = \frac{t^3(3t+2)}{2}$ ツテ

$S'(t) = 6t^3 + 3t^2$

$= 3t^2(2t+1)$

t	$\cdots$	$-\frac{3}{4}$	$\cdots$	$-\frac{1}{2}$	$\cdots$	0	$\cdots$
S'	$-$		$-$	0	$+$	0	$+$
S	↘		↘		↗		↗

$t < -\frac{3}{4}$において. 単調減少である. 1 ト

(Qのy座標) $q = -4t^3 - 3t^2 + 1$ より.

$T(t) = (-4t^3 - 3t^2 + 1 - 1) \times |t| \times \frac{1}{2}$

$= (-4t^3 - 3t^2) \times (-t) \times \frac{1}{2} = \frac{4t^4 + 3t^3}{2} = \frac{t^3(4t+3)}{2}$ ナ

$S(t) = T(t)$ をみたす t は.

$\frac{t^3(3t+2)}{2} = \frac{t^3(4t+3)}{2}$

$t^3(3t+2) = t^3(4t+3)$

$t^3(3t+2) - t^3(4t+3) = 0$

$t^3(3t+2-4t-3) = 0$

$t^3(-t-1) = 0$

$t \neq 0$ より.

$-t-1 = 0$

$t = -1$ ニヌ

3.

$a_n > 0$.

$a_n = a_1 \cdot r^{n-1}$

$a_1 \cdot a_3 = a_1 \cdot (a_1 \cdot r^2)$

$= a_1^2 \cdot r^2$

$a_4 = a_1 \cdot r^3$

$a_1 a_3 = a_4 = 16$ より

$a_1 r^3 = 16$.

$a_1 = \frac{16}{r^3}$

$a_1^2 \cdot r^2 = 16$ へ代入

$\left(\frac{16}{r^3}\right)^2 \cdot r^2 = 16$.

$\frac{\cancel{16^2}}{r^{\cancel{6}4}} \cdot \cancel{r^2} = \cancel{16}\,1$

$\frac{16}{r^4} = 1$.

$r^4 = 16 \quad r = \pm 2$.

$a_n > 0$ より $r = 2$.

$a_1 \cdot r^3 = 16$ より

$a_1 \times 2^3 = 16 \quad a_1 = 2$.

初項 $\underline{2}_{ア}$ 公比 $\underline{2}_{イ}$ より

$a_n = 2 \cdot 2^{n-1} = 2^n$.

$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} 2^k$

$= 2 \cdot \frac{2^n - 1}{2-1} = 2(2^n - 1)$

$= \underline{2^{n+1} - 2}_{ウエオ}$

$b_1 = 2, \quad b_{n+1} - b_n = 2n + 3$ より

$b_n = 2 + \sum_{k=1}^{n-1}(2k+3)$

$= 2 + \cancel{2} \cdot \frac{(n-1) \cdot n}{\cancel{2}} + 3(n-1)$

$= 2 + n^2 - n + 3n - 3$

$b_n = \underline{n^2 + 2n - 1}$ カ〜ク — (*)

$\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (k^2 + 2k - 1)$

$= \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} - n$.

$= \frac{n(n+1)(2n+1) + 6n(n+1) - 6n}{6}$

$= \frac{n\{(n+1)(2n+1) + 6(n+1) - 6\}}{6}$

$= \frac{n(2n^2 + 3n + 1 + 6n + 6 - 6)}{6}$

$= \underline{\frac{n(2n^2 + 9n + 1)}{6}}$ ケ〜ス.

$S_n = \sum_{k=1}^{n} a_k b_k$ とおく.

(*) より

$b_n = 2n^2 - (n^2 - 2n + 1)$

$= \underline{2}_{セ} n^2 - (n-1)^{\underline{2}_{ソ}}$

$f(n) = (n-1)^2 \cdot 2^n$ とおくと. $\quad f(n+1) = n^2 \cdot 2^{n+1}$

$a_n b_n = 2^n \cdot \{2n^2 - (n-1)^2\}$

$= 2^n \cdot 2n^2 - 2^n \cdot (n-1)^2$

$= n^2 \cdot 2^{n+1} - (n-1)^2 \cdot 2^n = f(n+1) - f(n)$

$S_n = \sum_{k=1}^{n} \{f(k+1) - f(k)\}$

$= \cancel{f(2)} - f(1) + \cancel{f(3)} - \cancel{f(2)} + \cancel{f(4)} - \cancel{f(3)} + \cdots + f(n+1) - \cancel{f(n)}$.

$= -f(1) + f(n+1)$

$= -(1-1)^2 \cdot 2^1 + n^2 \cdot 2^{n+1} = \underline{n^2 \cdot 2^{n+1}}$: 3 より

$S_1 = 1^2 \cdot 2^2 = 2^2 = 4$ (余り) 1

$S_2 = 2^2 \cdot 2^3 = 2^5 = 32$. (余り) 2

$S_3 = 3^2 \cdot 2^4 = 3^2 \cdot 2^4$ (余り) 0.

$S_4 = 4^2 \cdot 2^5 = (2^2)^2 \cdot 2^5$ (余り) $1^2 \cdot 2 = 2$.

$S_5 = 5^2 \cdot 2^6 = (3 \cdot 1 + 2)^2 \cdot 8^2$ (余り) $2^2 \cdot 2^2 = 4 \times 4 = 1 \times 1 = 1$

$S_6 = 6^2 \cdot 2^7$ (余り) 0

和 6

$S_7 = 7^2 \cdot 2^8 = (3 \cdot 2 + 1)^2 \cdot (2^2)^4$ (余り) $1^2 \cdot 1^4 = 1$.

⋮

$2015 \div 6 = 335 \cdots 5$.

5番目

和 6 が 335 組, と (1, 2, 0, 2, 1 / 0) ⇒ 6 が 336 コ.

$\sum_{k=1}^{2015} r_k = 6 \times 336 = \underline{2016}$ ツ〜ナ.

4.

$$\left[\begin{array}{ll} A(0,-1,0), & B(2,0,1) \\ C(0,0,-1), & D(3,2,1) \end{array}\right]$$

$\overrightarrow{AB} = -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(0,-1,0) + (2,0,1)$

$= (2,1,1)$ ア〜ウ.

$\overrightarrow{AC} = -\overrightarrow{OA} + \overrightarrow{OC}$

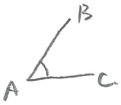
$= -(0,-1,0) + (0,0,-1)$

$= (0,1,-1)$ エ〜キ.

$|\overrightarrow{AB}| = \sqrt{2^2+1^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$ ク

$|\overrightarrow{AC}| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$ ケ.

$\cos\angle BAC = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\cdot|\overrightarrow{AC}|}$

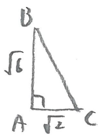


$= \dfrac{(2,1,1)\cdot(0,1,-1)}{\sqrt{6}\cdot\sqrt{2}}$

$= \dfrac{2\cdot 0 + 1\cdot 1 + 1\cdot(-1)}{2\sqrt{3}}$

$= \dfrac{1-1}{2\sqrt{3}} = 0.$

$\angle BAC = 90°$ コサ.



$S_{ABC} = \sqrt{2}\times\sqrt{6}\times\frac{1}{2} = \sqrt{3}$ シ.

$\overrightarrow{AB}\perp\vec{n}$, かつ, $\overrightarrow{AC}\perp\vec{n}$

$\Rightarrow \overrightarrow{AB}\cdot\vec{n} = \overrightarrow{AC}\cdot\vec{n} = 0$ ス であるから

$\vec{n} = (p, q, r)$ $(p \geqq 0)$ とおくと.

・$\overrightarrow{AB}\cdot\vec{n} = 0$ より

$(2,1,1)\cdot(p,q,r) = 0.$

$2p + q + r = 0$ —①

・$\overrightarrow{AC}\cdot\vec{n} = 0$ より

$(0,1,-1)\cdot(p,q,r) = 0$

$q - r = 0$ —②

②より $r = q$.

①へ代入

$2p + q + q = 0$

$2p + 2q = 0$

$p + q = 0$

$q = -p.$

$\vec{n} = (p, -p, -p)$

$= p(1,-1,-1)$ セ〜ツ.

単位ベクトル $\vec{n}$ より

$|\vec{n}| = 1$

$\sqrt{p^2+(-p)^2+(-p)^2} = 1.$

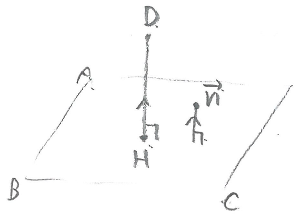
$\sqrt{p^2+p^2+p^2} = 1$

$3p^2 = 1$

$p^2 = \frac{1}{3}.$ $p = \pm\frac{\sqrt{3}}{3}$

$p \geqq 0$ より $p = \frac{\sqrt{3}}{3}.$

$\vec{n} = \frac{\sqrt{3}}{3}(1,-1,-1)$ テト



$\overrightarrow{DH} \parallel \vec{n}$ より

$\overrightarrow{DH} = t\vec{n}$

$\overrightarrow{DA} + \overrightarrow{AH} = t\vec{n}$ $\overrightarrow{AH} = \overrightarrow{AD} + t\vec{n}$

$\overrightarrow{AH}\perp\overrightarrow{DH}$ より $\overrightarrow{AH}\cdot\overrightarrow{DH} = 0 \Rightarrow \overrightarrow{AH}\cdot t\vec{n} = 0$

$(\overrightarrow{AD} + t\vec{n})\cdot t\vec{n} = 0$

$\overrightarrow{AD}\cdot t\vec{n} + t^2|\vec{n}|^2 = 0$

$\overrightarrow{AD} = -\overrightarrow{OA} + \overrightarrow{OD}$

$= -(0,-1,0) + (3,2,1)$

$= (3,3,1)$ より

$(3,3,1)\cdot\frac{\sqrt{3}}{3}t(1,-1,-1) + t^2|\vec{n}|^2 = 0$

$\frac{\sqrt{3}}{3}t(3-3-1) + t^2\cdot 1 = 0$

$-\frac{\sqrt{3}}{3}t + t^2 = 0.$

$t\left(t - \frac{\sqrt{3}}{3}\right) = 0.$

$t = 0,\ \frac{\sqrt{3}}{3}.$

$t \neq 0$ より. $t = \frac{\sqrt{3}}{3}$ ナニ.

したがって

$\overrightarrow{AH} = \overrightarrow{AD} + \frac{\sqrt{3}}{3}\vec{n}$

$= (3,3,1) + \frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{3}(1,-1,-1)$

$= \frac{1}{3}(9,9,3) + \frac{1}{3}(1,-1,-1)$

$= \frac{1}{3}(10,8,2)$

⇒(うら面)

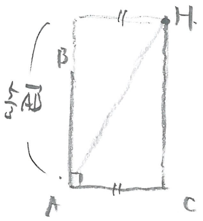
$\vec{AH} = \alpha\vec{AB} + \beta\vec{AC}$ とおくと.

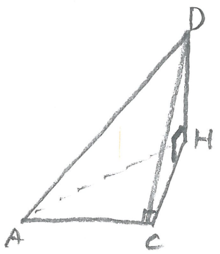
$\vec{AH} = \alpha(2,1,1) + \beta(0,1,-1)$

$= (2\alpha,\ \alpha+\beta,\ \alpha-\beta) = \frac{1}{3}(10, 8, 2)$

$$\begin{cases} 2\alpha = \frac{10}{3} & \Rightarrow \alpha = \frac{5}{3} \text{ また} \\ \alpha+\beta = \frac{8}{3} & \Rightarrow \beta = -\alpha + \frac{8}{3} = -\frac{5}{3} + \frac{8}{3} = \frac{3}{3} = 1 \\ \alpha-\beta = \frac{2}{3} & \end{cases}$$

$\vec{AH} = \frac{5}{3}\vec{AB} + \vec{AC}$





$|\vec{AC}| = \sqrt{2}$.

$|\vec{CH}| = \left|\frac{5}{3}\vec{AB}\right| = \frac{5}{3}|\vec{AB}| = \frac{5}{3} \times \sqrt{6} = \frac{5\sqrt{6}}{3}$.

$|\vec{DH}| = |t\vec{n}| = \frac{\sqrt{3}}{3}|\vec{n}| = \frac{\sqrt{3}}{3}$.

よって. 四面体ACDHの体積は.

$\sqrt{2} \times \frac{5\sqrt{6}}{3} \times \frac{1}{2} \times \frac{\sqrt{3}}{3} \times \frac{1}{3}$

$= \frac{2 \cdot 5 \cdot 3}{3 \cdot 2 \cdot 3 \cdot 3} = \frac{5}{9}$ //.